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Constrained dynamics of damped harmonic oscillator

Yen-chu Wang
Department of Electrical Engineering, Howard University, Washington, DC 20059, USA

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Abstract. A constrained dynamical formulation of the damped harmonic oscillator system has been obtained. The generalised classical Hamiltonian based on the Dirac theory and its quantal counterpart are given. With suitable gauge choices and perturbation technique, a quasiconstrained example shows that the wavepacket spreads and the uncertainty principle holds. The propagator is also derived. A discussion on the necessity and physical meaning of the constrained treatment is also included.

1. Introduction

Owing to its simplicity in calculation and in concept, a phenomenological approach to the problems of quantal dissipative systems can be considered as a viable alternative to the microscopic methods based on the density matrix. Whereas a large amount of literature exists (see, for example, Caldirola (1941) and Kanai (1948) on the time-dependent Hamiltonian, Kostin (1975) on the non-linear Hamiltonian and Hasse (1975) and Dekker (1981) for reviews), the phenomenological quantal dissipation problem has remained unsolved for more than forty years owing to the lack of a suitable Lagrangian and/or Hamiltonian self-consistently describing the quantum mechanics of a damped particle (see Brittin 1950, Greenberger 1979, Ray 1979, Cervero and Villaroel 1984).

In this paper a new treatment of the problem is presented. Instead of taking the system as a unconstrained dynamic system as previously perceived by all researchers working on this problem and as proved unsuccessful by others (Ray 1979), we consider it to be a constrained system described by a constrained generalised Hamiltonian. This fundamental change from the previous unconstrained approach is deemed necessary if one is to be consistent with the Hamiltonian formalism. In other words, if some of the interaction with its phenomenological surrounding medium or its internal structure is to be taken into account, the dynamics must be generalised to include constraints. Our generalised Hamiltonian is the mechanical energy which decays exponentially with time, consistent with the classical result. This Hamiltonian will be determined from the null total system energy Hamiltonian.

In § 2 we give the detailed derivation of the generalised Hamiltonian and its physical implications. In § 3 we present a short preliminary discussion of the quantisation of the problem based on perturbation. Section 4 presents the propagator, wavepacket amd uncertainty product as well as a discussion on the first-order solution of the Schrödinger equation. In § 5 we shall conclude our paper with remarks about other damped systems.

2. Generalised (constrained) Hamiltonian

For the underdamped harmonic oscillator Newton's equation and solution are

$$\ddot{x}+\gamma\dot{x}+\omega^2 x=0 \tag{2.1a}$$

$$x=x_0\exp(-\gamma t/2)\cos(\Omega_0 t+\alpha) \tag{2.1b}$$

where $\Omega_0=(\omega^2-\gamma^2/4)^{1/2}$ is the classical shifted frequency and α is an arbitrary phase. Multiplying by $m\dot{x}$ and integrating (2.1a) we have

$$\tfrac{1}{2}m\dot{x}^2+\tfrac{1}{2}m\omega^2x^2+m\gamma\int^t \dot{x}^2\,\mathrm{d}t=c. \tag{2.2}$$

Substituting (2.1b) into (2.2) causes the LHS to vanish. Thus the total system energy is always zero or the system Hamiltonian H_0 is always a null Hamiltonian. If canonical equations of motions are used, (2.2) with $p=m\dot{x}$ and $\mathrm{d}x=\dot{x}\,\mathrm{d}t$ gives

$$\dot{x}=p/m \tag{2.3a}$$

$$-\dot{p}=m\omega^2x+\gamma p \tag{2.3b}$$

provided that p is taken as an implicit function of x, i.e. that a constraint relation exists by use of the implicit function theorem. Although (2.3) correctly gives the equation of motion, it is purely formal because a null Hamiltonian implies constant x and p, which is obviously impossible. We therefore have to consider the motion being constrained with the constraint equation

$$\phi_1(x,p)=p^2/2m+m\omega^2x^2/2+\gamma\int^x p(s)\,\mathrm{d}s\approx 0 \tag{2.4}$$

where $\approx$ is Dirac's weak equality notation (see Dirac 1964). Since we are interested in the mechanical part of the energy another constraint equation involving mechanical energy is to be sought and can be derived from (2.4). Let us consider p in (2.4) as a function of x and differentiate (2.4) with respect to x to obtain

$$(p/m)\,\mathrm{d}p/\mathrm{d}x+m\omega^2x+\gamma p\approx 0 \tag{2.4a}$$

which can be integrated to give the following constraint equation:

$$\phi_2(x,p)=\Phi(x,p)\exp(\gamma\theta(x,p)/\Omega_0)-\varepsilon\approx 0 \tag{2.5}$$

where

$$\Phi=\frac{1}{2m}(p+\tfrac{1}{2}m\gamma x)^2+\tfrac{1}{2}m\Omega_0^2x^2 \qquad \Omega_0^2=\omega^2-\tfrac{1}{4}\gamma^2 \tag{2.6}$$

$$\theta=\tan^{-1}[m\Omega_0 x/(p+\tfrac{1}{2}m\gamma x)] \tag{2.7}$$

and ε is the integration constant specified as the mechanical energy at $x=0$. Without loss of generality we assume $x=0$ when $t=0$, and thus ε is the initial kinetic energy.

According to Dirac's constrained dynamics (Dirac 1950, 1958, 1964, Sudarshan and Mukunda 1974, Hansen *et al* 1976, Sundermeyer 1982), the generalised Hamiltonian of the system can be written as

$$H_T=H_0+\lambda_1\phi_1+\lambda_2\phi_2=\lambda_1\phi_1+\lambda_2\phi_2 \tag{2.8}$$

where $\lambda_{1,2}$ are Lagrangian multiplier functions and the equation of motion for any phase plane function $g(x, p)$ is given by

$$\dot{g} = \lambda_1\{g, \phi_1\} + \lambda_2\{g, \phi_2\} \tag{2.9}$$

where the brackets are Poisson brackets (PB) and the RHS is actually $\{g, H_T\}$. The consistency conditions require $\dot{\phi}_1 \approx 0$ and $\dot{\phi}_2 \approx 0$, which will characterise whether constraints $\phi_{1,2}$ are first class or not. Before doing that, we note

$$\partial\phi_1/\partial x = m\omega^2 x + \gamma p \qquad \partial\phi_1/\partial p = p/m \tag{2.10}$$

$$\partial\phi_2/\partial x = (m\omega^2 x + \gamma p)\exp(\gamma\theta/\Omega_0)$$

$$\partial\phi_2/\partial p = (p/m)\exp(\gamma\theta/\Omega_0) \tag{2.11}$$

so that

$$-\{\phi_2, \phi_1\} = \{\phi_1, \phi_2\} = \frac{\partial\phi_1}{\partial x}\frac{\partial\phi_2}{\partial p} - \frac{\partial\phi_1}{\partial p}\frac{\partial\phi_2}{\partial x} = 0. \tag{2.12}$$

Letting $g = \phi_{1,2}$ in (2.9) and using (2.12) we see that $\lambda_{1,2}$ remain arbitrary. (2.12) shows that $\phi_{1,2}$ are both first-class constraints and there is no second-class constraint. Since $\lambda_{1,2}$ are arbitrary H_T is not unique. We shall determine $\lambda_{1,2}$, and hence H_T, by setting two gauge conditions as follows. Letting $g = x$ and p in (2.9), respectively, we have

$$\dot{x} = \lambda_1\{x, \phi_1\} + \lambda_2\{x, \phi_2\} \tag{2.13a}$$

$$\dot{p} = \lambda_1\{p, \phi_1\} + \lambda_2\{p, \phi\} \tag{2.13b}$$

where

$$\{x, \phi_2\} = (p/m)\exp(\gamma\theta/\Omega_0) \tag{2.13c}$$

$$\{p, \phi_2\} = -(m\omega^2 x + \gamma p)\exp(\gamma\theta/\Omega_0). \tag{2.13d}$$

Comparing (2.13) with the equations of motion (2.3), we may choose

$$\lambda_1 = 0 \qquad \lambda_2 \equiv \lambda = \exp(-\gamma\theta/\Omega_0). \tag{2.14}$$

Dirac (1964) pointed out that the Lagrange multipliers must be functions of time (see also Lanczos 1970). This can be achieved from either the solution of the equation of motion or a constant of the motion. Gettys *et al* (1981) proved that

$$\theta = \Omega_0 t + \theta_0. \tag{2.15}$$

The constant θ_0 is unimportant and we shall set θ_0 zero. Thus (2.14) becomes

$$\lambda_1 = 0 \qquad \lambda_2 \equiv \lambda = \exp(-\gamma t) \approx \exp(-\gamma\theta/\Omega_0) \tag{2.16}$$

where the weak equality is used to mean that the last equality shall be used in the equations of motion only after the PB or any differentiation has been worked out.

Substituting (2.16) into (2.5) and (2.8), we have

$$H_T = \exp(-\gamma t)\Phi\exp(\gamma\theta/\Omega_0) - \exp(-\gamma t)\varepsilon \tag{2.17}$$

where Φ and θ are given by (2.6) and (2.7). It is noticed that our generalised Hamiltonian H_T is expressed in terms of the constraint obtained from the null Hamiltonian which is derived from the equation of motion. Therefore the constraint is of secondary type; there is no primary constraint here. The distinction between secondary and primary constraints becomes unimportant if one works out the constrained dynamics from the Hamiltonian rather than the Lagrangian (Dirac 1950). As far as the gauge condition is concerned, the special choice of $\lambda_1 = 0$ is made because any other choice would have to include ϕ_1 in the generalised Hamiltonian, which is difficult to work with. The physical state in any case does not depend on the gauge. We do not consider (2.15) as a constraint equation as it is independent of (2.4).

If one is also interested in the Lagrangian, one can obtain it from $L_T = p\dot{x} - \lambda_2\phi_2$ and $p = m\dot{x}$ as given by

$$L_T(x, \dot{x}) = m\dot{x}^2 - \exp(-\gamma t)\tfrac{1}{2}m[(\dot{x}+\tfrac{1}{2}\gamma x)^2 + \Omega_0^2 x^2]\exp\left[\frac{\gamma}{\Omega_0}\tan^{-1}\left(\frac{\Omega_0 x}{\dot{x}+\frac{1}{2}\gamma x}\right)\right] \tag{2.18}$$

where the term $\exp(-\gamma t)\varepsilon$ has been omitted as it does not affect the equation of motion in Lagrangian dynamics. It can be easily shown that the Lagrangian equation recovers (2.1*a*) when the weak equality rule for using (2.16) is followed.

The physical significance of the Lagrange multiplier and the auxiliary (constraint) conditions has been elegantly described by Lanczos (1970). He showed how to describe and account for the reaction of the system back to the source of the forces or to the external world. Most people are interested in constrained dynamics from the quantum field theory viewpoint; non-relativistic classical point mechanical constrained systems are usually considered to be of little physical relevance (Goldstein 1980). Our treatment of the damped oscillator as a constrained dynamic system is perhaps a counterexample if it is understood in the sense of Lanczos (1970).

It is interesting to elaborate on the implications of (2.17). Since the term $\exp(-\gamma t)\varepsilon$ is a purely time-dependent function and does not affect the equation of motion, (2.17) can be rewritten as

$$H = \lambda(t)\Phi\exp(\gamma\theta/\Omega_0) \qquad \lambda(t) = \exp(-\gamma t) \approx \exp(-\gamma\theta/\Omega_0). \tag{2.19}$$

The equation of motion (2.9) becomes simply

$$\dot{g} = \{g, H\}. \tag{2.20}$$

In (2.19) Φ is essentially the mechanical energy 'kernel' which is given by

$$\Phi = p^2/2m + \tfrac{1}{2}m\omega^2 x^2 + \tfrac{1}{4}\gamma(xp + px). \tag{2.21}$$

The last term represents the interaction between the mechanical system and the external world and it gives the energy lost to heat. The exponential factor being proportional to γ is a macroscopic description of the internal thermal motion of the phenomenological medium with which the particle interacts.

As far as the null Hamiltonian or constraint equation (2.4) is concerned, we want to point out that the integral is the integration of twice the Rayleigh dissipation function (Landau and Lifshitz 1980). Whereas they argued that a purely (unconstrained) mechanical treatment of a dissipative motion is impossible, they nevertheless foresaw that the motion would involve a previous history and hence integral operator. This has been made evident by the integral terms of (2.2) and (2.4).

3. Quantisation

Quantisation of constrained dynamics is notoriously difficult (Sundermeyer 1982) due to the perennial problem of operator ordering. We shall present a perturbation scheme guided by physical considerations. No sophisticated methods such as path integral quantisation (Faddeev and Slavnov 1980) or others (Nakamura and Mishima 1984, Sundermeyer 1982) will be discussed.

The Schrödinger equation is obtained from (2.19)

$$H\psi = \lambda\Phi \exp(\gamma\theta/\Omega_0)\psi = \mathrm{i}\hbar\, \partial\psi/\partial t. \tag{3.1}$$

In addition, supplementary conditions on ψ are (Dirac 1964)

$$\phi_1\psi = p^2\psi/2m + m\omega^2x^2\psi/2 + \gamma\left(\int^x p\, \mathrm{d}x\right)\psi = 0 \tag{3.2a}$$

$$\phi_2\psi = \Phi\exp(\gamma\theta/\Omega_0)\psi - \varepsilon\psi = 0. \tag{3.2b}$$

Let us assume Ω_0 is sufficiently large so that γ/Ω_0 is small and the time t is small so that γt is small. These approximations are physically interesting because the damping may actually be turned on for only a limited time period and γ/Ω_0 for practical quantal systems is usually small (underdamping). Thus we are interested in a quasiconstrained quantal system where damping acts for a short time. Then the exponential factor in (3.1) can be expanded as a power series in θ. We find it convenient to define the operator z to be the argument of the arctangent in (2.7) and use the series expansions (Abramowitz and Stegun 1964) for $1+z^2 \neq 0$ to obtain

$$\exp(\gamma\theta/\Omega_0) \simeq 1 + \delta z + \tfrac{1}{2}\delta^2 z^2 \tag{3.3}$$

where δ is the perturbation order parameter γ/Ω_0 and z is the symmetrised operator

$$z = \tfrac{1}{2}[(p+\tfrac{1}{2}m\gamma x)^{-1}m\Omega_0 x + m\Omega_0 x(p+\tfrac{1}{2}m\gamma x)^{-1}]. \tag{3.4}$$

If we retain terms up to the first order of δz in (3.3) then, after symmetrising the Hamiltonian in (3.1) and perturbing the wavefunction up to the first order of δ,

$$\psi = \psi_0 + \delta\psi_1 \tag{3.5}$$

the Schrödinger equation (3.1) becomes

$$\tfrac{1}{2}\lambda[(1+\delta z)\Phi + \Phi(1+\delta z)](\psi_0 + \delta\psi_1) = \mathrm{i}\hbar\partial(\psi_0+\delta\psi_1)/\partial t$$

$$\lambda\Phi\psi_0 = \mathrm{i}\hbar\partial\psi_0/\partial t \tag{3.6}$$

$$\lambda\Phi\psi_1 - \mathrm{i}\hbar\partial\psi_1/\partial t = -\eta(x,t) \tag{3.7}$$

$$\eta = \tfrac{1}{2}\lambda(z\Phi + \Phi z)\psi_0. \tag{3.8}$$

Let $K(x, x'; t, t')$ be the propagator of (3.6). Then

$$(\lambda\Phi - \mathrm{i}\hbar\partial/\partial t)K = -\mathrm{i}\hbar\delta(x-x')\delta(t-t') \tag{3.9}$$

$$\psi_1 = \frac{1}{\mathrm{i}\hbar}\iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}t'\, \eta(x',t')K(x,x';t,t'). \tag{3.10}$$

The meaning of (3.8) can be clarified as follows. Consider

$$(p+\tfrac{1}{2}m\gamma x)^{-1}\chi = F \tag{3.11}$$

where χ is a known function of x and t and F is an unknown function of x and t. Multiplying $(p+\tfrac{1}{2}m\gamma x)$ on both sides of (3.11) we have

$$\chi = (p+\tfrac{1}{2}m\gamma x)F = -\mathrm{i}\hbar\partial F/\partial x + \tfrac{1}{2}m\gamma F x.$$

Solving for the particular integral of F we get

$$F=\frac{\mathrm{i}}{\hbar}\exp(-\tfrac{1}{2}\alpha x^2)\int \exp(-\alpha x^2/2)\chi\,\mathrm{d}x \qquad \alpha=m\gamma/2\mathrm{i}\hbar. \tag{3.12}$$

Similarly let

$$(p+\tfrac{1}{2}m\gamma x)^{-1}x\chi+x(p+\tfrac{1}{2}m\gamma x)^{-1}\chi=\chi=2G[\chi(x,t)] \tag{3.13}$$

where we emphasise that G depends on the known function χ. Premultiplying $(p+\frac{1}{2}m\gamma x)$ and employing (3.11) and (3.12) we solve another first-order differential equation and obtain

$$G[\chi]=(1/\hbar)\exp(\tfrac{1}{2}\alpha x^2)\int^x \mathrm{d}s\exp(-\tfrac{1}{2}\alpha s^2)$$
$$\times\left[s\chi-\tfrac{1}{2}\exp(\tfrac{1}{2}\alpha s^2)\int^s \exp(-\tfrac{1}{2}\alpha\xi^2)\chi(\xi,t)\,\mathrm{d}\xi\right]. \tag{3.14}$$

Comparing (3.4) with (3.13) enables us to rewrite (3.8) as

$$\eta=\tfrac{1}{2}\lambda m\Omega_0\{G[\chi]_{\chi=\Phi\psi_0}+\Phi G[\chi]_{\chi=\psi_0}\}. \tag{3.15}$$

Along the same lines we can make the meaning of $(\int p\,\mathrm{d}x)\psi$ clear.

Let $(\int p\,\mathrm{d}x)\psi=I$ and take the partial derivative with respect to x. We get

$$p\psi+(I/\psi)\partial\psi/\partial x=\partial I/\partial x \qquad p=-\mathrm{i}\hbar\partial/\partial x \tag{3.16}$$

$$I=-\mathrm{i}\hbar\psi\ln\psi+c\psi \tag{3.16a}$$

where the last term is the homogeneous solution of I. Thus ($3.2a$) becomes

$$\phi_1\psi=-(\hbar^2/2m)\partial^2\psi/\partial x^2+\tfrac{1}{2}m\omega^2x^2\psi-\mathrm{i}\hbar\gamma\psi\ln\psi+\gamma c\psi=0. \tag{3.17}$$

The coefficient c, strictly speaking, is either a function of t or a complex constant. Here we may take it approximately as a slowly varying function of x. It can be determined as follows. We note that the operator $\int p\,\mathrm{d}x$ must be Hermitian and thus $I\psi^{-1}-I^*\psi^{*-1}=0$. From ($3.16a$) we obtain $c=\mathrm{i}\,\mathrm{Im}\,c=\mathrm{i}(\hbar/2)\ln(\psi^*\psi)$ where $\ln(\psi^*\psi)$ is assumed to be sufficiently smooth and slowly varying.

The supplementary conditions ($3.2b$) and (3.17) select, from the solutions (both physical and unphysical ones) of the Schrödinger equation, only the physical solutions, i.e. those which satisfy these conditions. The Hilbert space is said to be restricted to its physical part (Sundermeyer 1982, Shanmugadhasan 1963). As $\psi(x,t)$ depends on its initial wavefunction, say $\psi(x,0)$, not all functions will develop in time into admissible solutions to a general constrained quantal system. This is an important difference between constrained and unconstrained quantal systems. In a quasiconstrained quantal system as treated here, however, the restrictions imposed on $\psi(x,0)$, and hence on $\psi(x,t)$, are not as relevant as in the total constrained quantal systems. Another important difference between the two quantal systems is the mechanical energy operator ε. In constrained systems the mechanical energy operator and the Hamiltonian may be unequal. To see this let us compare (3.1) and ($3.2b$). Consistency between these two equations requires us to set $\varepsilon=\mathrm{i}\hbar\lambda^{-1}\partial/\partial t=\mathrm{i}\hbar\exp(\gamma t)\partial/\partial t$, a result not noted by Hasse (1975) who, treating dissipative systems as regular systems, correctly stated that the definition of an energy operator is still *open*, since the energy is not a constant of motion. For our system the constraint equation ($3.2b$) is proved to be equivalent to the Schrödinger equation (3.1) and it will no longer be considered as a supplementary condition on ψ. The only such condition is thus ($3.2a$) or (3.17).

If the gauge conditions were so chosen that $\lambda_1 = 1$ and $\lambda_2 = 0$ then the Hamiltonian would no longer be (2.17). From (2.8) it would become $H = \phi_1$ and the Schrödinger equation (see (3.17)) would be

$$-(\hbar^2/2m)\partial^2\psi/\partial x^2 + \tfrac{1}{2}m\omega^2 x^2\psi - \mathrm{i}\hbar\gamma\psi \ln\psi + \mathrm{i}(\hbar\gamma/2)\psi\ln(\psi^*\psi) = \mathrm{i}\hbar\partial\psi/\partial t \tag{3.18}$$

and the only supplementary condition on ψ would be (3.2*b*) $\phi_2\psi = 0$ where ε is the mechanical energy operator. It is easily shown that (3.18) likewise gives the Ehrenfest theorem (Schiff 1955)

$$\mathrm{d}\langle x\rangle/\mathrm{d}t = \langle p\rangle/m \qquad -\mathrm{d}\langle p\rangle/\mathrm{d}t = m\omega^2\langle x\rangle + \gamma\langle p\rangle. \tag{3.19}$$

It is interesting to note that the non-linear Schrödinger equation (3.18) is similar to that of Schuch *et al* (1983), obtained from a non-linear field theory and to that of Kostin (1975). For simplicity we shall not discuss the $\lambda_1 = 1$ gauge further except to point out that $\partial/\partial t$ in (3.18) should be zero as can be seen from (3.17) and the supplementary condition (3.2*b*) is the same as (3.1). Thus the two gauge choices ($\lambda_1 = 1, 0$) are physically equivalent (see Fradkin and Vilkovisky 1977).

Thus our system is governed by two Schrödinger equations (Dirac 1950) of which the first equation $\phi_1\psi = 0$ is time reversible since ϕ_1 is real (see (3.17)), implying the total system energy is conserved and the second equation $\phi_2\psi = 0$ is time irreversible since ϕ_2 is complex, implying the mechanical energy is not conserved.

4. Propagator

4.1. *Heisenberg operator solution*

The Green function can be obtained either from the Feynman path integral (Feynman and Hibbs 1965) or from the solution of the Heisenberg operator equation of motion (Landovitz *et al* 1983). We shall use the latter formalism for its practical calculation power. The theory of Landovitz *et al* is based on the fact that the basic operators x_H, p_H in the Heisenberg picture are linearly related to those in the Schrödinger picture through unknown time-dependent coefficients such as $a(t)$, $b(t)$, $c(t)$ and $d(t)$. Knowing these coefficients enables one to determine x_H and p_H which, in turn, enable one to find any observable (including the propagator for the Schrödinger equation) built up from x_H and p_H via unitary transformations. The advantage is that the coefficients are formally related to solutions of the classical equation of motion which we already know. In matrix form the coefficient matrix $R(t)$ is the solution of the first-order differential equation system

$$R(t) = M(t)R(t) \tag{4.1a}$$

$$R(0) = I \tag{4.1b}$$

and the following condition is required for $[x_\mathrm{H}, p_\mathrm{H}] = \mathrm{i}\hbar$:

$$ad - bc = 1 \tag{4.2}$$

where I is the identity matrix and

$$R(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad M(t) = \begin{pmatrix} J & F/m \\ -Gm\omega^2 & -J \end{pmatrix}. \tag{4.3}$$

On using (4.2) and (4.3), (4.1) becomes

$$\begin{pmatrix} \dot{a} & \dot{b} \\ \dot{c} & \dot{d} \end{pmatrix} = \exp(-\gamma t)\begin{pmatrix} \gamma/2 & m^{-1} \\ -m\omega^2 & -\gamma/2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4.4}$$

From the equation for $\dot{a}$ and $\dot{c}$ we can eliminate a and $\dot{a}$ to obtain

$$\ddot{c} + \gamma\dot{c} + \exp(-2\gamma t)\Omega_0^2 c = 0. \tag{4.5}$$

In a similar manner we obtain

$$\ddot{d} + \gamma\dot{d} + \exp(-2\gamma t)\Omega_0^2 d = 0. \tag{4.6}$$

Thus

$$c = \exp(-\gamma t/2)(A_1 \sin\zeta + B_1 \cos\zeta)\zeta^{-1/2} \tag{4.7}$$

$$d = \exp(-\gamma t/2)[A_2 \sin\zeta + B_2 \cos\zeta]\zeta^{-1/2} \tag{4.8}$$

where $\zeta = (\Omega_0/\gamma)\exp(-\gamma t)$.

It can be shown from (4.4) that

$$a = -(\exp(\gamma t)\dot{c} + \gamma c/2)/m\omega^2 \qquad b = -(\exp(\gamma t)\dot{d} + \gamma d/2)/m\omega^2. \tag{4.9}$$

The unknown constants A_1 and B_1 can be determined from $c(0) = 0$ and $a(0) = 1$ as a consequence of (4.1b). Similarly A_2 and B_2 can be determined from $d(0) = 1$ and $b(0) = 0$. For $\delta = \gamma/\Omega_0$ small we have

$$a = d \simeq \cos\alpha \qquad \alpha \equiv \alpha(t) = \delta^{-1} - \zeta = (1 - \exp(-\gamma t))\Omega_0/\gamma \tag{4.10}$$

$$b \simeq (m\Omega_0)^{-1}\sin\alpha \qquad c \simeq -(m\omega^2/\Omega_0)\sin\alpha. \tag{4.11}$$

Equation (4.2) can be verified by substitution of the above equations. Landovitz *et al* (1983) have shown that the propagator $K(x, x'; t)$ is given by

$$K = f(t)\exp[\mathrm{i}g(\mathrm{d}x^2 + ax'^2 - 2xx')] \tag{4.12}$$

$$f(t) = (g/\mathrm{i}\pi)^{1/2} \tag{4.13}$$

$$g = 1/2\hbar b. \tag{4.14}$$

Substituting (4.10) and (4.11) into the above we have

$$K(x, x'; t) = f(t)\exp[\mathrm{i}g(t)(x^2\cos\alpha + x'^2\cos\alpha - 2xx')] \tag{4.15}$$

$$f(t) = (m\Omega_0/2\mathrm{i}\hbar\pi\sin\alpha)^{1/2} \qquad g(t) = m\Omega_0/2\hbar\sin\alpha. \tag{4.16}$$

In comparison with the propagator of the undamped harmonic oscillator, we see that the propagator here is different only in α. As $\gamma t \to 0$, $\alpha \to \omega t$ and they become the same.

4.2. *Wavepacket*

Consider the Gaussian wavepacket

$$\psi_0(x', 0) = N\exp(\mathrm{i}k_0 x' - x'^2/4a^2) \qquad N^{-1} = (2\pi a^2)^{1/4}. \tag{4.17}$$

It will evolve in time according to

$$\psi_0(x, t) = \int_{-\infty}^{\infty} \mathrm{d}x'\, \psi_0(x', 0)K(x, x'; t). \tag{4.18}$$

When (4.15)-(4.17) are substituted into (4.18), then with the help of

$$\int_{-\infty}^{\infty} \mathrm{d}y \exp(-ay^2+by) = \left(\frac{\pi}{a}\right)^{1/2} \exp(b^2/4a) \tag{4.19}$$

we have

$$\psi_0(x, t) = W \exp(-Ax^2 + Bx) \tag{4.20}$$

$$W = f(t)(2a)^{1/2}(2\pi)^{1/4}(1+\mathrm{i}4ga^2 \cos \alpha)^{1/2}\Delta^{-1} \exp[-(k_0 a/\Delta)^2 - \mathrm{i}4k_0^2 a^4 g\Delta^{-2} \cos \alpha] \tag{4.21}$$

$$A = (2ga/\Delta)^2 + \mathrm{i}(16g^3a^4\Delta^{-2} \cos \alpha - g \cos \alpha) = A_\mathrm{r} + \mathrm{i}A_\mathrm{i} \tag{4.22}$$

$$B = 4k_0 ga^2\Delta^{-2} + \mathrm{i}16k_0 g^2 a^4 \Delta^{-2} \cos \alpha = B_\mathrm{r} + \mathrm{i}B_\mathrm{i} \tag{4.23}$$

$$\Delta^2 = 1 + (4a^2 g \cos \alpha)^2. \tag{4.24}$$

Since ψ_0 is normalised W can be shown to satisfy

$$\psi_0^*\psi_0 = |W|^2 \exp(-2A_\mathrm{r}x^2 + 2B_\mathrm{r}x) \tag{4.25a}$$

$$|W|^2(\pi/2A_\mathrm{r})^{1/2} \exp(B_\mathrm{r}^2/2A_\mathrm{r}) = 1. \tag{4.25b}$$

In order to calculate the expectation values we need

$$\int_{-\infty}^{\infty} \mathrm{d}y\, y \exp(-ay^2+by) = \left(\frac{\pi}{a}\right)^{1/2} \frac{b}{2a} \exp(b^2/4a) \tag{4.26a}$$

$$\int_{-\infty}^{\infty} \mathrm{d}y\, y^2 \exp(-ay^2+by) = \left(\frac{\pi}{a}\right)^{1/2} \frac{1}{2a}\left(1+\frac{b^2}{2a}\right) \exp(b^2/4a). \tag{4.26b}$$

On using (4.25) and (4.26) the expectation values of x and x^2 are given by

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi_0^* x \psi_0 \, \mathrm{d}x = B_\mathrm{r}/2A_\mathrm{r} \tag{4.27a}$$

$$\langle x^2 \rangle = (1 + B_\mathrm{r}^2/A_\mathrm{r})/4A_\mathrm{r}. \tag{4.27b}$$

Using (4.25), (4.26) and the identities

$$\partial\psi_0/\partial x = \psi_0(-2Ax + B) \qquad \partial^2\psi_0/\partial x^2 = (B^2 - 2A)\psi_0 - 4ABx\psi_0 + 4A^2x^2\psi_0 \tag{4.28}$$

we have

$$\langle p \rangle = \hbar(B_\mathrm{i} - A_\mathrm{i}B_\mathrm{r}/A_\mathrm{r}) \qquad \langle p^2 \rangle = \langle p \rangle^2 + \hbar^2(A_\mathrm{r} + A_\mathrm{i}^2/A_\mathrm{r}). \tag{4.29}$$

Since $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$ and $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$, (4.27) and (4.29) give

$$\Delta x \Delta p = (\hbar/2)[1 + (A_\mathrm{i}/A_\mathrm{r})^2]^{1/2}. \tag{4.30}$$

Substituting (4.22)-(4.24) into (4.27), (4.29) and (4.30) we obtain

$$\langle x \rangle = \hbar k_0 \sin \alpha / m\Omega_0 \qquad \langle x^2 \rangle = (\hbar \sin \alpha / 2ma\Omega_0)^2 + (a \cos \alpha)^2 + (\hbar k_0 \sin \alpha / m\Omega_0)^2 \tag{4.31}$$

$$\begin{aligned} \langle p \rangle &= \hbar k_0 \cos \alpha \\ \langle p^2 \rangle &= (\hbar k_0 \cos \alpha)^2 + \{4a^4(\hbar m\Omega_0)^2 + [4a^4(m\Omega_0)^2 - \hbar^2]^2 \sin^2 \alpha \cos^2 \alpha\} \\ &\quad \times \{4a^2[\hbar^2 \sin^2 \alpha + 4a^4(m\Omega_0)^2 \cos^2 \alpha]\}^{-1} \end{aligned} \tag{4.32}$$

$$\Delta x \Delta p = (\hbar/2)\{1 + [(2a^2 m\Omega_0)^2 - \hbar^2]^2(\sin \alpha \cos \alpha)^2/(2a^2 m\Omega_0\hbar)^2\}^{1/2}. \tag{4.33}$$

The expectation value of $\langle xp+px\rangle$ can be easily shown to be $-i\hbar - i2\hbar(-2A\langle x^2\rangle + B\langle x\rangle)$ which becomes, upon using (4.27), $-2\hbar(2A_i\langle x^2\rangle - B_i\langle x\rangle)$ or

$$\langle xp+px\rangle = -2\hbar \cos\alpha \sin\alpha[4a^4(m\Omega_0)^2 - \hbar^2(1+k_0^2 4a^2)]/4a^2 m\hbar\Omega_0. \tag{4.34}$$

Equation (4.33) shows that the packet is a minimum uncertainty wavepacket at $t=0$ but pulsates afterwards only for finite time before becoming minimum again. For comparisons with the strangled harmonic oscillator see Colegrave and Kherabady (1984). From (4.25a)

$$|\psi|^2_{t>0} = |W|^2 \exp(k_0^2 A_r/2g^2)\exp[-(x-k_0/4g)^2/2a'^2]$$

$$2a'^2 = (2A_r)^{-1}$$

and from (4.17)

$$|\psi_0|^2_{t=0} = \exp(-x^2/2a^2)/(2\pi a^2)^{1/2}.$$

Thus its centre pulsates in time and is located at $k_0/4g = \hbar k_0 \sin\alpha/2m\Omega_0$ starting at $x=0$ when $t=0$. Its width is a at $t=0$ but spreads in time periodically since the width for $t>0$ is

$$a' = a[\cos^2\alpha + (\hbar \sin\alpha/2m\Omega_0 a^2)^2]^{1/2}$$

with a non-classical frequency $\alpha/t = \Omega_0[1-\exp(-\gamma t)]/\gamma t$. All these features are quite different from those obtained with unconstrained dynamics (see Hasse 1975).

4.3. Discussions on ψ_1

For ψ_0 as shown in (4.20) the function $G[\chi]$ in (3.14) can be evaluated by using the integral (Abramowitz and Stegun 1975)

$$\int dx \exp[-(ax^2+2bx+c)] = \frac{1}{2}\left(\frac{\pi}{a}\right)^{1/2} \exp[(b^2-ac)/a]\,\mathrm{erf}(a^{1/2}x + b/a^{1/2})$$

$$\mathrm{erf}\, z = 2\pi^{-1/2}\exp(-z^2)\sum_{k=0}^{\infty} \frac{2^k}{1\times 3\times\ldots\times(2k+1)} z^{2k+1}$$

and series expansions similar to the above, obtained by repeated integrations by parts. For the calculation of $G[\Phi\psi_0]$, (4.28) can conveniently be used. Substitution of (3.15) into (3.10) on using (4.15) and (4.16) where we make the change $t \to t-t'$ (see also (4.10)) enables us to calculate ψ_1 by integration. It is suggested that the space integration be carried out first as the time integration is more complicated. For the case where $\gamma t \to 0$ with $\gamma \neq 0$ the Hamiltonian becomes time independent and the calculation of ψ_1 can be simplified.

5. Conclusion

A self-consistent constrained Hamiltonian formalism for an underdamped harmonic oscillator has been formulated. Its quantisation scheme as a constrained quantal dynamic system with suitable gauge conditions (the $\lambda_1 = 0$ gauge) has been presented. We have shown that the Hamiltonian can no longer be considered as a regular unconstrained one; it must instead be generalised in order to account for the interaction between the damped oscillator and its medium. We have taken a quasiconstrained case as an example, based on a perturbation technique, to show that the Gaussian wavepacket spreads in time periodically together with an oscillating centre and that the Heisenberg uncertainty principle holds without any controversy.

Our theory is sufficiently general for further investigations of other physically interesting quantities such as transition amplitudes, transition elements, scattering matrix elements and coherent states. Our theory can be expanded to the system of many damped oscillators. Its generalisation to other types of damped particle systems such as the damped free particle is straightforward. In this case (2.4) and (2.4*a*) still hold provided we set $\omega = 0$. Equation (2.5), however, becomes $\phi_2 = \Phi - \varepsilon \approx 0$ where $\Phi = (p + m\gamma x)^2/2m$. For the damped forced particle there is no null Hamiltonian and $H_0 \neq 0$ as the potential is attractive and the total system energy no longer zero. The first equality in (2.8), however, remains true. $\phi_{1,2}$ can also be similarly obtained. These and other interesting systems will be presented in the future.

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